Part I — Foundations of Linear Systems

System of Equations to Matrices

$$ax + by = c, \qquad dx + ey = f$$

$$ax + by = \begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = c, \qquad dx + ey = \begin{pmatrix} d \\ e \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = f$$

$$\Rightarrow \quad \begin{matrix} ax + by = c \\ dx + ey = f \end{matrix} \quad \Rightarrow \quad \begin{matrix} \begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = c \\ \begin{pmatrix} d \\ e \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = f \end{matrix} \quad \Rightarrow \quad \begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix}.$$

$$\Rightarrow \quad \text{[linear combination or span]} \quad \begin{pmatrix} a \\ c \end{pmatrix} x + \begin{pmatrix} b \\ e \end{pmatrix} y = \text{span}\left\{ \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} b \\ e \end{pmatrix} \right\} = \begin{bmatrix} c \\ f \end{bmatrix}.$$

Span: "In general" the span means to just multiple the objects in the set by arbitrary constants and add them together. E.g., $\text{span}\{e^{rt}, e^{-rt}\} = c_1 e^{rt} + c_2 e^{-rt} = y_h$ [homogeneous solution to an ODE (span of set of solutions in a linear algebra + differential equations course)].

The column vectors are $\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ e \end{pmatrix}$. The row vectors are $\begin{bmatrix} a & b \end{bmatrix}, \begin{bmatrix} c & e \end{bmatrix}$.

If you solve this, e.g., (for example)

$$x - y = 1, \qquad x + y = 2$$

The solution to the augmented matrix will indicate whether the lines intersect, are parallel, or lie atop each other.

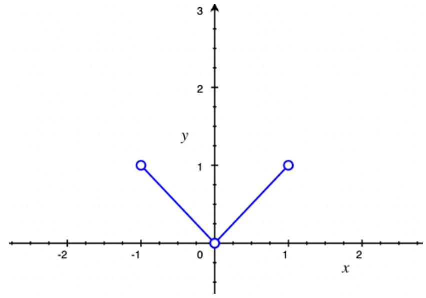
$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \end{pmatrix}.$$

This question could be stated as "Is (1,2) in the span of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$?" "Is $\mathbf{b}$ in the span of $\mathbf{v}_1, \mathbf{v}_2$?"

The span $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} x + \begin{pmatrix} -1 \\ 1 \end{pmatrix} y = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Yes, there is a unique or infinite (consistent) solution. So, yes $\mathbf{b} = (1,2)$ is in the span of $\{\mathbf{v}_1, \mathbf{v}_2\}$.

NOTE: In Linear Algebra, you will do the exact arithmetic over and over. The question will just be asked differently, such as "Solve the system." or "Is a vector in the span?" ...

These are the column vectors $\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}$



This is **not the span** of these vectors.
The span can go infinitely in both directions i.e., $\pm$ which forms a plane.

Since the solution 'exists', it is in the span.
It is in the span of whether it is unique or infinite.

The Vector Notation

$x, \quad Ax = b$	$\vec{x}, \quad A\vec{x} = \vec{b}$	$\mathbf{x}, \quad A\mathbf{x} = \mathbf{b}$
$x = (x_1, x_2, \dots)$	$\vec{x} = (x_1, x_2, \dots) = \begin{pmatrix}x_1\\x_2\\\vdots\end{pmatrix}$	$\mathbf{x} = (x_1, x_2, \dots) = \begin{bmatrix}x_1\\x_2\\\vdots\end{bmatrix}$
Gilbert Strang	**Universal**	**David Lay** *(universal)*

PHYISICS & Trig: $\vec{A} = \langle x, y, z \rangle$, $P_0(x_0, y_0, z_0)$

3D CALC: $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, $P_0(x_0, y_0, z_0)$, $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

Vector Calculus or Linear Algebra: $\mathbf{v} = (v_1, v_2, \dots)$ same as a point (x_0, y_0), $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T \mathbf{v}$.

NOTE* 3D calculus and basic physics are usually stuck in three dimensions, so we stick with x, y, z. Whereas 'Vector Calculus', i.e., Linear Algebra—The Study of Vector Space, is in nth dimensional space. So, we move into indices because the alphabet stops at 26 letters (right now).

Column Vectors [Quantum Mechanics Vetor(s)]

$$x = \vec{x} = \mathbf{x} = (x_1, x_2, \dots) = [x_1, x_2, \dots \rangle(\text{Griffeths}) = \begin{bmatrix}x_1\\x_2\\\vdots\end{bmatrix} = \begin{pmatrix}x_1\\x_2\\\vdots\end{pmatrix}$$

Row Vector

$$\mathbf{x}^T = \langle x_1, x_2, \dots](\text{Griffeths}) = \begin{bmatrix}x_1\\x_2\\\vdots\end{bmatrix}^T = \begin{bmatrix}x_1 & x_2 & \cdots\end{bmatrix} = \{x_1, x_2, \dots\} \text{ (depends)}$$

E.g., in Wolfram Alpha, a matrix of row vectors $\{\{1,2,3\},\{2,3,1\}\} = \begin{bmatrix}1 & 2 & 3\\2 & 3 & 1\end{bmatrix}$.

If you spanned the set $\mathcal{W} = \left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}3\\1\end{pmatrix}\right\}$, you would get span $\mathcal{W} = \begin{bmatrix}1 & 2 & 3\\2 & 3 & 1\end{bmatrix}\mathbf{x}$.

The above are all the same thing; they just change author-to-author and book-to-book.

